THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 53, No. 2, February 1974
Printed in U.S.A.

Losses and Impulse Response in Parabolic Index Fibers With Square Cross Section

By D. MARCUSE

(Manuscript received August 13, 1973)

Mode coupling is studied in a parabolic index fiber with a lossy boundary and square cross section. Statistical deviations of the fiber axis from perfect straightness and random changes of its width are considered as causing mode coupling. The excess loss caused by these mode coupling mechanisms and the loss penalty incurred for a certain degree of narrowing of the impulse response are estimated.

I. INTRODUCTION

Multimode optical fibers whose cores have parabolic distributions of the refractive index,[1–3]

$$n = n_0\left(1 - \frac{r^2}{a^2}\,\bar{\Delta}\right), \tag{1}$$

are of great practical interest for light transmission over long distances, since their delay distortion is much less serious than that of conventional clad fibers.

Since no optical fiber can ever be produced free of random imperfections, it is important to know how statistical irregularities of the fiber affect its performance. Random irregularities of the fiber axis and random changes of the effective width of the fiber cause coupling among its modes. The mode losses are functions of the mode number. Absorption losses tend to affect all modes in the same way. However, if we assume that the fiber boundary either consists of an absorptive material or is a rough surface that scatters light, we must expect that higher order modes, whose fields reach strongly into the neighborhood of the fiber boundary, suffer much higher losses than lower order modes that are confined to the vicinity of the fiber axis. Coupling of the low-order modes to the high-loss, high-order modes increases the overall waveguide losses. One objective of our study of the effect of waveguide irregularities is thus the determination of the excess losses caused by mode coupling.

The second objective of this study of waveguide irregularities consists in determining the impulse response of the fiber. In the absence of coupling, each mode transports a fraction of the total power at its characteristic group velocity. Since the group velocities of different mode groups are not identical, pulse distortion results.[2,3] Mode coupling has the beneficial effect of improving the impulse response of the fiber. It is thus of interest to determine how much reduction of multimode pulse distortion can be achieved by random bends and random width changes of the fiber.

The effect of random bends on parabolic index fibers with circular cross section has been estimated in an earlier paper.[4] Pure diameter changes of a fiber with circular cross section leave modes with different circumferential symmetries uncoupled. Statistical irregularities are unlikely to result in pure diameter changes without distorting the circular fiber cross section. However, an analysis of more general distortions of a fiber with nominally circular cross section is difficult to perform. For this reason we discuss a fiber with parabolic index distribution (1) but with square cross section. It seems reasonable to expect that the performance of a fiber with square cross section is similar to that of a fiber with circular cross section. We expect to find the correct order of magnitude of the losses and impulse response of the round fiber by examining its close relative, the fiber with square cross section. In particular, it should be possible to assess the relative effect of random axis deformations as compared to random width changes. In a square fiber, changes of only one set of opposing walls leave groups of modes uncoupled from each other. This situation corresponds to the circular fiber with pure diameter changes that leave modes of different azimuthal symmetry uncoupled. By allowing both sets of opposing walls to change their separation randomly, we are sure that all modes are coupled to each other. This model corresponds to a nominally round fiber whose cross section is deformed in an arbitrary way that does not conserve the circular symmetry.

II. THE MODES OF THE PERFECT FIBER WITH SQUARE CROSS SECTION

The modes of an infinitely extended medium with the distribution

$$n^2 = n_0^2\left(1 - 2\,\frac{r^2}{a^2}\,\bar{\Delta}\right) \tag{2}$$

of the square of the refractive index have the form[5]

$$E_{pq} = \frac{2\left(\sqrt{\frac{\mu_0}{\epsilon_0}}\,P\right)^{\frac{1}{2}} H_p\left(\sqrt{2}\,\frac{x}{w}\right) H_q\left(\sqrt{2}\,\frac{y}{w}\right) e^{-r^2/w^2}}{(n_0\pi 2^{p+q} p\,!\,q\,!)^{\frac{1}{2}} w}\, e^{-i\beta z}. \tag{3}$$

It is

$$r^2 = x^2 + y^2. \tag{4}$$

The parameter a is an arbitrary constant that, in conjunction with $\bar{\Delta}$, determines the transverse dependence of the refractive index distribution. However, in the round fiber it is convenient to associate a with the radius of the fiber boundary so that $\bar{\Delta}$ is the relative difference between the values of the refractive index on axis and at the fiber boundary.

The square of the refractive index distribution (2) does not follow precisely from (1). However, if one equation is regarded as the precise distribution of the corresponding quantity, the other holds approximately provided $\bar{\Delta}$ is small and we limit r to the range $r \leqq a$. H_p and H_q are Hermite polynomials of degree p and q, and P is the power carried by the mode. The parameter ω is defined as[5] ($k = \omega\sqrt{\epsilon_0\mu_0}$)

$$w = \left(\frac{\sqrt{2}a}{n_0 k\sqrt{\bar{\Delta}}}\right)^{\frac{1}{2}} \tag{5}$$

and determines the radius of the field distribution with $p = q = 0$. At $r = w$ the field has decayed to $1/e$ of its value on axis. E_{pq} represents the transverse component of the electric field vector. The longitudinal field components are relatively much weaker and are not being considered. The field (3) is only an approximate solution of Maxwell's equations. The propagation constant of the mode is given as[5]

$$\beta = \beta_{pq} = n_0 k\left[1 - \frac{2\sqrt{2\bar{\Delta}}}{n_0 k a}\,(p + q + 1)\right]^{\frac{1}{2}}. \tag{6}$$

The modes of the square-law medium are mutually orthogonal and satisfy the relation

$$\frac{\beta}{2k}\sqrt{\frac{\epsilon_0}{\mu_0}}\int_{-\infty}^{\infty} E_{pq}E^*_{p'q'}\,dx\,dy = P\delta_{pp'}\delta_{qq'}. \tag{7}$$

So far, the fiber boundary has been ignored. The mode field (3) is an (approximate) solution of the guided-wave problem if we assume that the distribution (2) extends to infinity. However, each mode decays very rapidly outside of a certain region. For a given value of p the field oscillates as a function of x passing through p zero crossings. The shape of the function

$$H_4\left(\sqrt{2}\,\frac{x}{w}\right)e^{-x^2/w^2} \tag{8}$$

is shown in Fig. 1. At the point (see appendix)

$$x = x' = w\sqrt{p + \tfrac{1}{2}} \tag{9}$$

the oscillatory behavior of the function changes to a rapid decay. If $x' < a$ the presence of the wall does not interfere appreciably with the field distribution. However, if $x' > a$ the field distribution is severely altered by the presence of the wall. Since we are assuming that the interaction of the field with the wall causes power dissipation either by absorption or by radiation, we consider those modes whose fields reach the vicinity of the wall with high field intensity as being effectively cut off. By replacing x' in (9) with a we obtain the condition for the maximum value of p that can be allowed for low-loss modes.

$$p_c = \left(\frac{a}{w}\right)^2 - \frac{1}{2} = n_0 ka\sqrt{\frac{\bar{\Delta}}{2}} - \frac{1}{2}. \tag{10}$$

Since we are assuming that the boundary of the fiber has a square cross

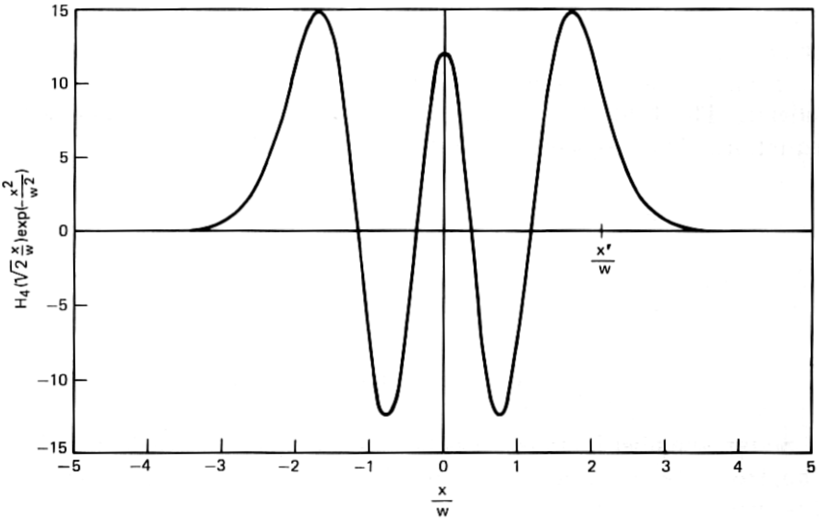

Fig. 1—Plot of the function given by eq. (8).

section, we also must impose the same "cutoff" condition on q,

$$q_c = \left(\frac{a}{w}\right)^2 - \frac{1}{2}. \tag{11}$$

We use the modes (3) of the infinite square-law medium (2) to describe the modes of the fiber with square boundary if $p < p_c$ and $q < q_c$. If either $p > p_c$ and/or $q > q_c$ we regard the modes as so lossy that they are effectively cut off. This procedure is an approximation, but it allows us to obtain estimates (whose errors are unknown) to a complicated problem.

III. COUPLING COEFFICIENTS

The coupling coefficients between two modes are defined by the general expression[6,7]

$$K_{pq,p'q'} = \frac{\omega\epsilon_0}{4iP}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\,(\bar{n}^2 - n^2)E_{pq}E_{p'q'}^{*}. \tag{12}$$

A square-law fiber with random axis deformations can be described by the following distribution of the square of its refractive index.

$$\bar{n}^2 = n_0^2\left\{1 - \frac{2}{a^2}[(x-f)^2 + (y-g)^2]\bar{\Delta}\right\}. \tag{13}$$

We assume that f and g are both random functions of z and that f/a and g/a are small quantities. The deflection of the optical axis of the square-law medium has a far more important effect on the modes than the corresponding deviation of the fiber boundary. The deflection of the fiber boundary that results from the random bends of its axis is neglected.

Substitution of (2), (3), and (13) into (12) results in

$$K_{pq,p\pm 1,q} = \frac{n_0 k w \bar{\Delta}}{ia^2}\sqrt{p + {}^{1}_{0}}\, f(z) \tag{14}$$

and

$$K_{pq,p,q\pm 1} = \frac{n_0 k w \bar{\Delta}}{ia^2}\sqrt{q + {}^{1}_{0}}\, g(z). \tag{15}$$

All other coupling coefficients vanish. The two choices (1 or 0) that are indicated under the square-root sign in (14) and (15) belong to the corresponding upper or lower sign of the subscript on the left-hand side. Random deformations of the fiber axis couple only neighboring modes.

For random width changes of the fiber we use the distribution

$$\bar{n}^2 = n_0^2 \left\{1 - 2\left[\frac{x^2}{(a+f)^2} + \frac{y^2}{(a+g)^2}\right]\bar{\Delta}\right\}. \tag{16}$$

For small values of f/a and g/a we can write this expression approximately as follows:

$$\bar{n}^2 = n_0^2 \left\{1 - \frac{2}{a^2}\left[x^2\left(1 - 2\frac{f}{a}\right) + y^2\left(1 - 2\frac{g}{a}\right)\right]\bar{\Delta}\right\}. \tag{17}$$

For random changes of the width of the guide we obtain from (2), (3), (12), and (17)

$$K_{pq,p\pm2,q} = \frac{n_0 k w^2 \bar{\Delta}}{2ia^3}\sqrt{(p \pm 1)(p + \substack{2\\0})}\, f(z) \tag{18}$$

and

$$K_{pq,p,q\pm2} = \frac{n_0 k w^2 \bar{\Delta}}{2ia^3}\sqrt{(q \pm 1)(q + \substack{2\\0})}\, g(z). \tag{19}$$

All other coupling coefficients vanish. There are nonvanishing diagonal elements in this case. However, diagonal elements couple each mode only to itself. This self-coupling is of no importance if $f(z)$ and $g(z)$ have Fourier spectra with no zero (spatial) frequency component.

IV. COUPLED POWER THEORY

Mode coupling in waveguides with random irregularities can be described by coupled power equations.[8]

$$\frac{\partial P_\nu}{\partial z} + \frac{1}{v_\nu}\frac{\partial P_\nu}{\partial t} = -\alpha_\nu P_\nu + \sum_{\mu=1}^{N} h_{\nu\mu}(P_\mu - P_\nu). \tag{20}$$

P_ν is the average power carried by the mode labeled ν, v_ν is its group velocity, and α_ν its power loss coefficient. The mode label ν is used as an abbreviation for the set of labels p, q. The power coupling coefficient $h_{\nu\mu}$ is defined as follows:[8]

$$h_{\nu\mu} = |\hat{K}_{\nu\mu}|F(\beta_\nu - \beta_\mu). \tag{21}$$

The coefficient $\hat{K}_{\nu\mu}$ is the factor of the function $f(z)$ or $g(z)$ appearing in eqs. (14), (15), (18), and (19). The spatial power spectrum of the function $f(z)$ [or $g(z)$] is defined as

$$F(\theta) = \frac{1}{L}\left\langle\left|\int_0^L f(z)e^{-i\theta z}dz\right|^2\right\rangle. \tag{22}$$

It is assumed that $L \to \infty$ in (22). The symbol $\langle\ \rangle$ indicates an ensemble average.

Since the random processes considered here tend to couple each mode only to one of its neighbors on either side (in mode label space) the equation system (20) can be converted to a partial differential equation whose variables are not only the length coordinate z and time t but in addition the two mode labels p and q.[9,10] If the number of modes below the effective cutoff value is very large, the set of discrete modes can be regarded as a quasicontinuum. We write

$$\sum_{p',q'} h_{pq,p'q'}(P_{p'q'} - P_{pq})$$
$$= h_{pq,p+\Delta p,q}(P_{p+\Delta p,q} - P_{pq}) + h_{pq,p-\Delta p,q}(P_{p-\Delta p,q} - P_{pq})$$
$$+ h_{pq,p,q+\Delta q}(P_{p,q+\Delta q} - P_{pq}) + h_{pq,p,q-\Delta q}(P_{p,q-\Delta q} - P_{pq})$$
$$\approx (\Delta p)^2 \frac{\partial}{\partial p}\left[h(p)\frac{\partial P}{\partial p}\right] + (\Delta q)^2 \frac{\partial}{\partial q}\left[h(q)\frac{\partial P}{\partial q}\right]. \tag{23}$$

The last step follows by considering the discrete mode labels as continuous variables and replacing differences by differentials. The notation $h(p)$ and $h(q)$ serves as a reminder that, according to (14) and (15), the coupling coefficients depend only on p if q is held fixed, and (18) and (19) show that they depend only on q if p is held fixed. We thus obtain the approximate partial differential equation

$$\frac{\partial P}{\partial z} + \frac{1}{v}\frac{\partial P}{\partial t} = +\ (\Delta p)^2 \frac{\partial}{\partial p}\left[h(p)\frac{\partial P}{\partial p}\right] + (\Delta q)^2 \frac{\partial}{\partial q}\left[h(q)\frac{\partial P}{\partial q}\right]. \tag{24}$$

The average mode power P is now regarded as a continuous function of z, t, p, and q. The group velocity v is a function of p and q. We have omitted the loss term. We consider the modes as lossless if the variables p and q remain below the cutoff values (10) and (11) and as having infinitely high loss if cutoff is exceeded. This fact can be incorporated into the theory as a boundary condition by requiring

$$P(p_c, q_c) = 0. \tag{25}$$

It has been shown in Ref. 8 how the pulse propagation problem can be solved by means of a perturbation method if the solutions of (24) for the time-independent case are known. We thus consider the trial solution

$$P(z, t, p, q) = U(p)V(q)e^{-\sigma z} \tag{26}$$

and obtain by substitution into (24)

$$(\Delta p)^2 \frac{1}{U} \frac{\partial}{\partial p} \left[h(p) \frac{\partial U}{\partial p} \right] + (\Delta q)^2 \frac{1}{V} \frac{\partial}{\partial q} \left[h(q) \frac{\partial V}{\partial q} \right] + \sigma = 0. \quad (27)$$

We separate this equation into two ordinary differential equations by introducing the separation constant κ^2:

$$\frac{d}{dp} \left[h(p) \frac{dU}{dp} \right] + \frac{\kappa^2}{(\Delta p)^2} U = 0 \quad (28)$$

and

$$\frac{d}{dq} \left[h(q) \frac{dV}{dq} \right] + \frac{\sigma - \kappa^2}{(\Delta q)^2} V = 0. \quad (29)$$

V. CALCULATION OF THE STEADY-STATE POWER LOSS

The equation system (28) and (29) together with the boundary condition (25) (and an additional one to be discussed later) defines an eigenvalue problem. The lowest order eigenvalue σ_{11} has the physical meaning of the steady-state loss of the statistical power distribution.[8] This quantity is of interest since it determines the additional losses that are caused by the statistical irregularities of the fiber.

For random deformations of the fiber axis we obtain the power coupling coefficient $h(p)$ from (14) and (21).

$$h(p) = K(\Omega)p \quad (30)$$

with

$$K(\Omega) = \left(\frac{n_0 k w \bar{\Delta}}{a^2} \right)^2 F(\Omega). \quad (31)$$

We assume that $f(z)$ and $g(z)$ have identical power spectra so that $h(q)$ follows from $h(p)$ by replacing p with q. According to (6) the difference of the propagation constants of adjacent modes can be approximated as

$$\beta_{p+1,q} - \beta_{pq} = \Omega \approx \frac{\sqrt{2\bar{\Delta}}}{a}. \quad (32)$$

This approximation is independent of the mode numbers. This means that only one spatial frequency (or actually a very narrow range of spatial frequencies) of the power spectrum $F(\Omega)$ is responsible for mode coupling. For random axis deformations we have

$$\Delta p = \Delta q = 1 \quad (33)$$

so that we must solve the differential equation

$$\frac{d}{dp}\left[p\,\frac{dU}{dp}\right] + \frac{\kappa^2}{K(\Omega)}\,U = 0. \tag{34}$$

Its solution is a Bessel function of zero order,

$$U(p) = J_0\left(2\,\frac{\kappa}{\sqrt{K(\Omega)}}\,\sqrt{p}\right). \tag{35}$$

The choice of the Bessel function instead of a Neumann function, that would also solve (34), is dictated by an additional boundary condition. Since the partial differential equation (24) can be regarded as a diffusion process, we must require that no power diffuses into the lowest order mode $p = 0$ from negative values of p. This requirement means that $\partial P/\partial p = 0$ at $p = 0$. The solution (35) satisfies this condition. The solution of (29) is similarly

$$V(q) = J_0\left(2\sqrt{\frac{\sigma - \kappa^2}{K(\Omega)}}\,\sqrt{q}\right). \tag{36}$$

The boundary condition (25) leads to

$$2\,\frac{\kappa}{\sqrt{K(\Omega)}}\,\sqrt{p_c} = u_\nu \tag{37}$$

and

$$2\sqrt{\frac{\sigma - \kappa^2}{K(\Omega)}}\,\sqrt{q_c} = u_\mu. \tag{38}$$

The roots u_ν and u_μ are defined as solutions of the equation

$$J_0(u_\nu) = 0. \tag{39}$$

Since the eigenvalues depend on the labels ν and μ, we attach these labels to σ and obtain from (37) and (38) (note, $p_c = q_c$)

$$\sigma_{\nu\mu} = \frac{K(\Omega)}{4p_c}\,(u_\nu^2 + u_\mu^2). \tag{40}$$

The steady-state power loss, the lowest order eigenvalue σ_{11}, follows from (5), (10) (neglecting the term $\frac{1}{2}$), (31), and $u_1 = 2.405$

$$\text{axis deformation:}\quad \sigma_{11} = 5.78\,\frac{\bar{\Delta}}{a^4}\,F(\Omega). \tag{41}$$

We have thus rederived the loss formula (42) of Ref. 4.

For random diameter changes we obtain from (18), (21), and (31), considering that the spacing (in β-space) between adjacent coupled modes is now twice as large,

$$h(p) = \frac{1}{4}\frac{w^2}{a^2}K(2\Omega)p^2. \tag{42}$$

Since the number of modes is assumed to be large, we have used the approximation $p(p-1) \approx p^2$. With $\Delta p = \Delta q = 2$ we obtain from (28) and (42)

$$\frac{d}{dp}\left[p^2\frac{dU}{dp}\right] + \frac{a^2}{w^2}\frac{\kappa^2}{K(2\Omega)}U = 0. \tag{43}$$

The solution of this differential equation is

$$U = \frac{1}{\sqrt{p}}\cos\left(\rho_\nu \ln p + \phi_\nu\right) \tag{44}$$

with

$$\rho_\nu = \sqrt{\frac{a^2}{w^2}\frac{\kappa^2}{K(2\Omega)} - \frac{1}{4}}. \tag{45}$$

The solution of (29) is correspondingly

$$V = \frac{1}{\sqrt{q}}\cos\left(\rho_\mu \ln q + \phi_\mu\right) \tag{46}$$

with

$$\rho_\mu = \sqrt{\frac{a^2}{w^2}\frac{\sigma - \kappa^2}{K(2\Omega)} - \frac{1}{4}}. \tag{47}$$

These solutions have a singular behavior at $p = 0$ or $q = 0$. However, we must keep in mind that p and q are really discrete quantities. Considering them as continuous variables is an approximate procedure that can work only for very large values of p or q where the relative difference between adjacent discrete values becomes small. Since $\ln p = 0$ for $p = 1$, we allow p and q to vary only between 1 and p_c. The requirement that no power diffuses across the lower limit of the range of the variables imposes the conditions

$$\left(\frac{dU}{dp}\right)_{p=1} = 0$$

and

$$\left(\frac{dV}{dq}\right)_{q=1} = 0. \tag{48}$$

These conditions lead to the determination of the phase terms via the equations

$$\tan \phi_\nu = -\frac{1}{2\rho_\nu} \tag{49}$$

and

$$\tan \phi_\mu = -\frac{1}{2\rho_\mu}. \tag{50}$$

The boundary condition (25) leads to

$$\rho_\nu = \frac{1}{\ln p_c}\left[(2\nu - 1)\frac{\pi}{2} + \arctan\frac{1}{2\rho_\nu}\right]. \tag{51}$$

ρ_ν as well as ρ_μ are solutions of this equation with integer values of ν. Since the values of ρ_ν are now known, we obtain the eigenvalue $\sigma_{\nu\mu}$ from (45) and (47)

$$\sigma_{\nu\mu} = \frac{w^2}{a^2} K(2\Omega)(\rho_\nu^2 + \rho_\mu^2 + \tfrac{1}{2}). \tag{52}$$

For the lowest order eigenvalue, that is, for the steady-state power loss coefficient, we obtain with the help of (5) and (31)

$$\text{width changes:} \quad \sigma_{11} = (4\rho_1^2 + 1)\frac{\bar{\Delta}}{a^4} F(2\Omega). \tag{53}$$

The solution of (51) is not a constant. It depends on the waveguide parameters through its dependence on $p_c = (a/w)^2$. A plot of ρ_1 as a function of p_c is shown in Fig. 2.

The forms of the steady-state power loss coefficients (41) and (53) are very similar. The power spectra describe the deflection of the fiber axis from its nominally straight position or the changes of the width of one of the transverse fiber dimensions. However, the excess loss caused by random changes of the width of the fiber depends on a component of the power spectrum at twice the spatial frequency compared to the excess loss for random bends of the fiber axis.

VI. CALCULATION OF THE PULSE WIDTH

The width of the impulse response of a multimode fiber that is long enough for the steady-state distribution to establish itself is given by the formula:[8]

$$\Delta t = 4\left\{L \sum_{\nu,\mu}^{N} \frac{(G_{11}, VG_{\nu\mu})^2}{\sigma_{\nu\mu} - \sigma_{11}}\right\}^{\frac{1}{2}}. \tag{54}$$

The term with $\sigma_{\nu\mu} = \sigma_{11}$ is excluded from the sum. The functions

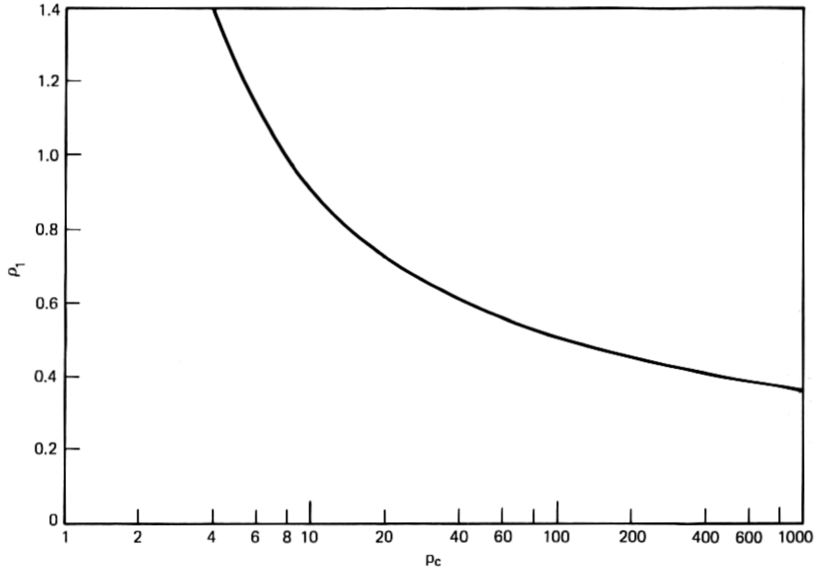

Fig. 2—Plot of the parameter ρ_1 as a function of p_c.

$G_{\nu\mu}$ are defined as

$$G_{\nu\mu} = A_{\nu\mu} U_\nu(p) V_\mu(q) \tag{55}$$

and V is by definition the difference between the inverse group velocity of the modes minus the inverse of the maximum group velocity.[4]

$$V = \frac{1}{v(p,\, q)} - \frac{n_0}{c} = \frac{\bar{\Delta}}{cn_0k^2a^2}\,(p + q)^2. \tag{56}$$

The expression in parenthesis is an abbreviated way of writing

$$(G_{11},\, VG_{\nu\mu}) = \int dp \int dq G_{11} V G_{\nu\mu}. \tag{57}$$

The integrals extend over the entire range of p and q variables from either 0 or 1 to the cutoff value $p_c = q_c$. Requiring the normalization

$$\int dp \int dq G_{\nu\mu}^2 = 1, \tag{58}$$

we have for random axis deformations:

$$G_{\nu\mu} = \frac{w^2}{a^2}\,\frac{J_0\left(u_\nu\sqrt{\dfrac{p}{p_c}}\right) J_0\left(u_\mu\sqrt{\dfrac{q}{q_c}}\right)}{J_1(u_\nu) J_1(u_\mu)} \tag{59}$$

and for random width changes:

$$G_{\nu\mu} = \frac{A_{\nu\mu}}{\sqrt{pq}} \cos[\rho_\nu \ln p + \phi_\nu] \cos[\rho_\mu \ln q + \phi_\mu] \tag{60}$$

with

$$A_{\nu\mu} = 2\left\{\left[\ln p_c + \frac{2}{1+4\rho_\nu^2}\right]\left[\ln q_c + \frac{2}{1+4\rho_\mu^2}\right]\right\}^{-\frac{1}{2}}. \tag{61}$$

The integrals (57) have the following solutions. For axis deformations ($\mu \neq 1$):

$$\begin{aligned}(G_{11}, VG_{1\mu}) &= (G_{11}, VG_{\mu 1}) \\ &= \frac{32\bar{\Delta} p_c^2 u_1 u_\mu}{cn_0 k^2 a^2 (u_1^2 - u_\mu^2)^2}\left\{\frac{2u_1^2 - 1}{3u_1^2} - \frac{6(u_1^2 + u_\mu^2)}{(u_1^2 - u_\mu^2)^2}\right\}\end{aligned} \tag{62}$$

and for ν, $\mu \neq 1$

$$(G_{11}, VG_{\nu\mu}) = \frac{2^7 \bar{\Delta}}{cn_0 k^2 a^2} \frac{u_1 u_\nu}{(u_1^2 - u_\nu^2)^2} \frac{u_1 u_\mu}{(u_1^2 - u_\mu^2)^2} p_c^2. \tag{63}$$

For random width changes ($\mu \neq 1$):

$$\begin{aligned}(G_{11}, VG_{1\mu}) = (G_{11}, VG_{\mu 1}) &= -4 \frac{\bar{\Delta}}{cn_0 k^2 a^2} \rho_1 \rho_\mu p_c^2 A_{11} A_{1\mu} \\ &\times \left\{ \frac{2\rho_1^2\left(1 - \frac{1}{p_c}\right)}{(1+4\rho_1^2)[1+(\rho_1+\rho_\mu)^2][1+(\rho_1-\rho_\mu)^2]} \right. \\ &\times \left[(-1)^\mu + \frac{1 + 2(\rho_1^2 + \rho_\mu^2)}{p_c\sqrt{(1+4\rho_1^2)(1+4\rho_\mu^2)}}\right] \\ &+ \frac{1}{A_{11}[4+(\rho_1+\rho_\mu)^2][4+(\rho_1-\rho_\mu)^2]} \\ &\left. \times \left[(-1)^\mu + \frac{5 + 2(\rho_1^2 + \rho_\mu^2)}{p_c^2\sqrt{(1+4\rho_1^2)(1+4\rho_\mu^2)}}\right]\right\}\end{aligned} \tag{64}$$

and for ν, $\mu \neq 1$

$$\begin{aligned}(G_{11}, VG_{\nu\mu}) &= 8 \frac{\bar{\Delta}}{cn_0 k^2 a^2} p_c^2 A_{11} A_{\nu\mu} \\ &\times \frac{\rho_1^2 \rho_\nu \rho_\mu}{[1+(\rho_1+\rho_\nu)^2][1+(\rho_1-\rho_\nu)^2][1+(\rho_1+\rho_\mu)^2][1+(\rho_1-\rho_\mu)^2]} \\ &\times \left\{(-1)^\nu + \frac{1 + 2(\rho_1^2 + \rho_\nu^2)}{p_c\sqrt{(1+4\rho_1^2)(1+4\rho_\nu^2)}}\right\} \\ &\times \left\{(-1)^\mu + \frac{1 + 2(\rho_1^2 + \rho_\mu^2)}{p_c\sqrt{(1+4\rho_1^2)(1+4\rho_\mu^2)}}\right\}.\end{aligned} \tag{65}$$

Evaluation of (54) with the help of (62) and (63) yields, for random deformations of the fiber axis,

$$\Delta t = \frac{0.42\bar{\Delta}p_c^2\sqrt{L}}{cn_0k^2a^2\sqrt{K(\Omega)}}\frac{a}{w}. \tag{66}$$

Equation (66) determines the pulse width of an impulse after it has traveled a distance L (L must be large enough so that the pulse has settled down to steady state) in the presence of random deformations of the fiber axis. The pulse width for uncoupled modes is obtained from (56)

$$\Delta T = \frac{L}{v(p_c, q_c)} - \frac{L}{v(0, 0)} = \frac{4\bar{\Delta}L}{cn_0k^2a^2}p_c^2. \tag{67}$$

The relative improvement of the width of the impulse response caused by mode coupling is characterized by the ratio[8]

$$R = \frac{\Delta t}{\Delta T} = \frac{0.105}{\sqrt{LK(\Omega)}}\frac{a}{w}. \tag{68}$$

Mode coupling not only shortens the width of the impulse response, but it also leads to excess loss. In order to find out how much excess loss is associated with a given improvement of the width of the impulse response, we form the product of (41) with the square of (68)

$$R^2\sigma_{11}L = 0.032. \tag{69}$$

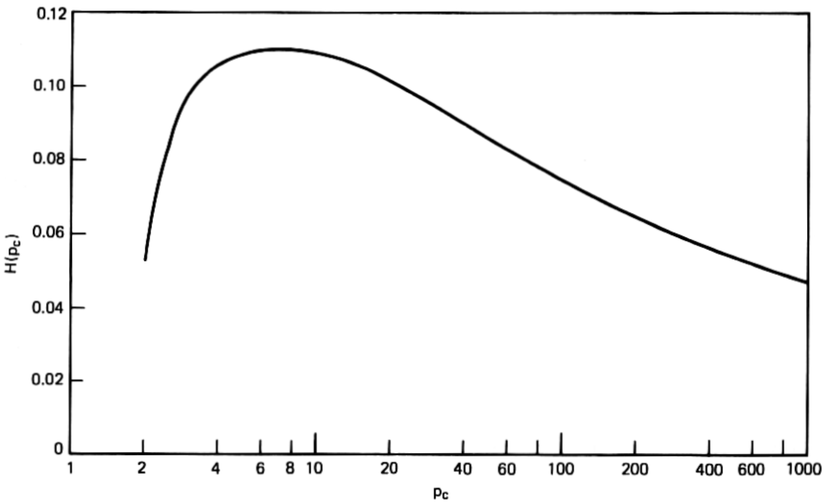

Fig. 3—Plot of the function $H(p_c)$.

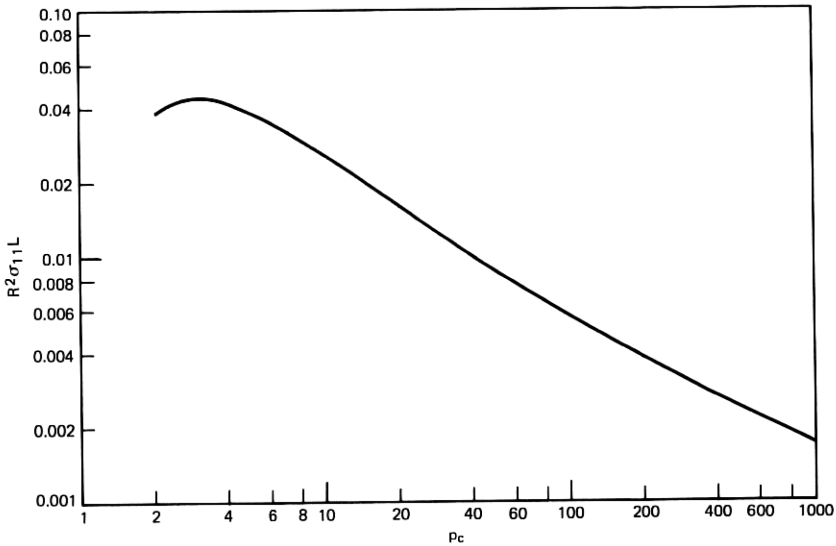

Fig. 4—Plot of the loss penalty as a function of p_c.

For random axis deformations, the product of the square of the improvement factor with the excess loss is independent of the waveguide parameters and the statistics of the random axis deformations.

For random width changes we obtain similarly

$$\Delta t = \frac{4\bar{\Delta} p_c^2}{c n_0 k^2 a^2} \sqrt{\frac{L}{K(2\Omega)}}\, H(p_c)\, \frac{a}{w}. \tag{70}$$

The function $H(p_c)$ is plotted in Fig. 3. The improvement factor is

$$R = \frac{H(p_c)}{\sqrt{LK(2\Omega)}} \frac{a}{w}. \tag{71}$$

Finally, we obtain the loss penalty from

$$R^2\sigma_{11}L = (2\rho_1^2 + 0.5)H^2(p_c). \tag{72}$$

This function is shown graphically in Fig. 4.

VII. DISCUSSION AND NUMERICAL RESULTS

We have derived expressions for the steady-state loss and the loss penalty of graded index fibers with square cross section for the case of random axis deformation and random width changes. The most conspicuous difference between these two types of fiber imperfections is the fact that, whereas the Fourier components of the function $f(z)$ (describing the fiber axis) at the spatial frequency Ω are instrumental in the mode mixing process, the Fourier components at twice the spatial frequency, 2Ω, determine the mode mixing process in case of random changes of the width of the fiber. This behavior can easily be understood. Consider a Gaussian beam of arbitrary width that is injected into the fiber off axis.[11] The beam undulates periodically around the fiber axis and also changes its width periodically. The undulations around the fiber axis have a period[11]

$$\Lambda = \frac{\sqrt{2}\pi a}{\sqrt{\bar{\Delta}}} \tag{73}$$

while the width changes repeat themselves with half that period or at twice the spatial frequency.[11] Random displacements of the fiber axis couple to the deflections of the beam from its on-axis position. This deflection is driven by a Fourier component at the spatial frequency

$$\Omega = \frac{2\pi}{\Lambda} = \frac{\sqrt{2\bar{\Delta}}}{a}. \tag{74}$$

The beam width changes are correspondingly driven by changes in the gradient (width) of the parabolic index medium. It is thus clear that they respond to twice the spatial frequency.

In order to be able to associate specific rms deviations of the fiber axis or rms width changes with fiber loss we have to consider a particular statistical model. We choose (arbitrarily) a Gaussian correlation function

$$\langle f(z) f(z+u) \rangle = \bar{\sigma}^2 e^{-(u/D)^2}. \tag{75}$$

$\bar{\sigma}$ is the rms deviation of the function $f(z)$ and D its correlation length. The power spectrum of $f(z)$ is known to be[12]

$$F(\theta) = \sqrt{\pi}\bar{\sigma}^2 D e^{-(\theta D/2)^2}. \tag{76}$$

For a given value of θ this function assumes its maximum value

$$[F(\theta)]_{D=D_m} = \frac{\sqrt{2\pi} e^{-0.5}}{\theta} \bar{\sigma}^2 = \frac{1.52}{\theta} \bar{\sigma}^2 \tag{77}$$

at

$$D_m = \sqrt{2}/\theta. \tag{78}$$

Let us consider a numerical example. We use the following fiber parameters

$$\left.\begin{aligned} a &= 4.85 \times 10^{-3}\ \text{cm} \\ n_0 &= 1.56 \\ \bar{\Delta} &= 0.014 \end{aligned}\right\}. \tag{79}$$

At $\lambda = 1\ \mu$m wavelength we have $a/w = 6.3$ or $p_c = 39.7$. The difference between the propagation constant of adjacent modes is, according to (32), $\Omega = 34.5\ \text{cm}^{-1}$. With $\theta = \Omega$ we calculate the excess loss values at the peak of the power spectrum at the value of the correlation length given by (78), $D_m = 0.041$ cm. For random deviations of the waveguide axis we obtain from (41), (77), and (79) ($\bar{\sigma}$ in cm)

$$\sigma_{11} = 6.44 \times 10^6 \bar{\sigma}^2\ (\text{cm}^{-1}). \tag{80}$$

In order to keep the excess loss below 10 dB/km $= 2.3\ 10^{-5}\ \text{cm}^{-1}$ we must keep the rms deviation of the fiber axis below $\bar{\sigma} = 2 \times 10^{-6}$ cm. However, this stringent tolerance requirement results from our assumption that the correlation length of the random irregularities of the fiber axis assumes its worst possible value (78). If, for example, the correlation length happens to be $D = 0.5$ cm we obtain instead of (80)

$$\sigma_{11} = 6.4 \times 10^{-25} \bar{\sigma}^2\ (\text{cm}^{-1}) \tag{81}$$

so that we can now tolerate $\bar{\sigma} = 6 \times 10^9$ cm in order to keep the excess loss below 10 dB/km. This example shows that it is impossible to predict the excess loss to be expected from a practical square-law fiber unless the statistics of its irregularities are known precisely.

For reasons of comparison we state the corresponding value for random width changes. In this case the spatial frequency that is instrumental in providing mode coupling is $2\Omega = 69\ \text{cm}^{-1}$. The worst possible correlation length is now $D_m = 0.02$ cm. From (53) and Fig. 2 with $p_c = 40$ we obtain ($\bar{\sigma}$ in cm)

$$\sigma_{11} = 1.39 \times 10^6 \bar{\sigma}^2\ (\text{cm}^{-1}). \tag{82}$$

The tolerance requirements of random width changes appear a little less stringent than those of random axis deformations. However, we have already pointed out that the excess loss value depends critically on the actual statistics of the fiber. Since the excess loss caused by random axis deviations depends on a different spatial frequency than the excess loss caused by random width changes, a loss comparison of the two effects is not possible.

Next we discuss the loss penalty that is incurred for a given improvement of the width of the impulse response of coupled mode operation compared to uncoupled mode operation. The equations (69) and (72) show that the loss penalty is independent of the statistics of the fiber irregularities. This feature makes the loss penalty a useful quantity. In case of random variations of the fiber axis, the loss penalty is even independent of the fiber parameters and is simply a dimensionless number. Let us assume that we want to achieve a ten-fold improvement of the width of the impulse response compared to the impulse response of uncoupled multimode operation. In this case we have $R = 0.1$ and obtain from (69) for random deviations of the fiber axis

$$\sigma_{11}L = 3.2 = 14 \text{ dB}. \tag{83}$$

The length L needed to incur this loss and at the same time to achieve $R = 0.1$ depends on the statistics of the irregularities. However, eq. (83) tells us that it costs 14 dB in excess loss to achieve a ten-fold relative pulse width improvement.

For random width changes, the situation is slightly different. Here the loss penalty depends somewhat on the fiber parameters. For the values used earlier we find from Fig. 4 with $p_c = 40$,

$$R^2\sigma_{11}L = 10^{-2}. \tag{84}$$

The loss penalty for $R = 0.1$ is more favorable in this case, $\sigma_{11}L = 4.34$ dB.

Fiber irregularities can be introduced intentionally in order to improve the impulse response. In the conventional fiber with a round core of constant refractive index that is surrounded by a cladding with constant index, the loss penalty for pulse distortion improvement can be reduced (in principle avoided) by tailoring the shape of the power spectrum carefully.[8] The reason that the shape of the power spectrum has an influence on the loss penalty is explained by the observation that the spacing (in β-space) between adjacent modes of the conventional fiber is dependent on the mode number, so that a band of spatial frequencies of the power spectrum takes part in the mode coupling process.

In case of the parabolic index fiber, only one spatial frequency (or actually a narrow range of spatial frequencies) is responsible for mode coupling. The shape of the power spectrum is thus immaterial, only its value at the spatial frequency Ω enters into the picture. The expressions (69) and (72) show that the loss penalty of the parabolic index fiber is independent of the power spectrum. No loss advantage is to be gained by using especially shaped power spectra. One might think that an advantage could be gained by departing from the square-law index profile in order to change the mode spacing and sample more of the power spectrum. But as soon as the index distribution deviates slightly from the parabolic profile the uncoupled impulse response becomes much broader. The mode coupling mechanism would now have to work against a far less favorable (uncoupled) impulse response so that it seems unlikely that an advantage can be gained in this way.

Finally, we consider an example of pulse width reduction by random irregularities. We can introduce intentional deviations of the fiber axis from perfect straightness in order to cause mode coupling. Since the coupling process must be random, we could use deformation functions $f(z)$ and $g(z)$ that are sinusoidal in shape but have a random phase. This introduces a power spectrum centered around the spatial frequency of the sinusoidal process having a finite width. Instead of pursuing this idea further, we assume that we have somehow created an axis deformation whose power spectrum reaches beyond the frequency Ω of (32). For simplicity, and to have a definite case in mind, we choose

$$F(\theta) = \begin{cases} \dfrac{\pi\bar{\sigma}^2}{2\Omega} & |\theta| < 2\Omega \\ 0 & |\theta| > 2\Omega. \end{cases} \tag{85}$$

This power spectrum is flat from zero spatial frequencies to a cutoff value of $\theta = 2\Omega$ and zero for $\theta > 2\Omega$. The rms deviation $\bar{\sigma}$ of the fiber axis from a straight line appears in (85). Using the fiber parameters (79) we obtain from (68) ($\bar{\sigma}$, L in cm)

$$R = \frac{6.9 \times 10^{-5}}{\bar{\sigma}\sqrt{L}}. \tag{86}$$

A ten-fold improvement of the width of the impulse response (compared to the uncoupled case), $R = 0.1$, over a length of $L = 1$ km $= 10^5$ cm requires an rms deviation of the fiber axis of $\bar{\sigma} = 2.2 \times 10^{-6}$ cm. We already know that we pay for this improvement of the impulse response with a loss penalty of 14 dB. Very slight random deviations from perfect straightness are already very effective in providing mode coupling and improving the width of the impulse response.

For random width changes we have to allow for a wider power spectrum. Letting the power spectrum again extend twice as far as the effective spatial frequency, 2Ω in this case, forces us to divide (85) by 2. We thus find from Fig. 3 and (71)

$$R = \frac{8.37 \times 10^{-5}}{\bar{\sigma}\sqrt{L}}. \tag{87}$$

$R = 0.1$ and $L = 10^5$ cm requires $\bar{\sigma} = 2.6 \times 10^{-6}$ cm.

APPENDIX

The function

$$H_p\left(\sqrt{2}\,\frac{x}{w}\right) e^{-x^2/w^2} e^{-i\beta_p z} \tag{88}$$

describes the modes of a square-law medium defined by

$$n(x) = n_0\left(1 - \frac{x^2}{a^2}\,\bar{\Delta}\right). \tag{89}$$

The associated ray problem can be described by a paraxial Hamiltonian of the form[13]

$$H = \frac{p_x^2}{2n_0} - n(x). \tag{90}$$

The quantum mechanical treatment of this problem leads to an expression for the "energy" E of the ray that has the form[14]

$$E = -\frac{\beta_p}{k}. \tag{91}$$

We define the "turning point" of the light rays associated with the wave field (88) by the condition that p_x, which is proportional to the slope of the light ray, must vanish. That means that the ray trajectory is tangential to the optical axis as the rays turn back in their path leading them away from the axis. Using $p_x = 0$ and equating (90) and (91) we find the following condition for the turning point:

$$n(x') = \frac{\beta_p}{k}. \tag{92}$$

The propagation constant of this two-dimensional mode field is[5]

$$\beta_p = n_0 k\left[1 - 2\,\frac{\sqrt{2\bar{\Delta}}}{n_0 k a}\left(p + \frac{1}{2}\right)\right]^{\frac{1}{2}}. \tag{93}$$

Substitution of (89) and (93) into (92) leads with the help of (5) to the formula (9) for the turning point.

The physical argument advanced here serves the purpose of defining the range in which the Hermite polynomial has an oscillatory behavior. This range is given by

$$-x' \leqq x \leqq x'. \tag{94}$$

Outside of this range the Hermite polynomial grows monotonically to infinite values. However, since the Hermite polynomial enters the mode field (88) only as a product with a Gaussian function, the mode field decays rapidly without oscillation outside of the range given by (94).

REFERENCES

1. S. E. Miller, U.S. Patent No. 3,434,774, "Waveguide for Millimeter and Optical Waves," issued March 25, 1969.
2. D. Gloge and E. A. J. Marcatili, "Multimode Theory of Graded-Core Fibers," B.S.T.J., *52*, No. 9 (November 1973), pp. 1563–1578.
3. D. Marcuse, "The Impulse Response of an Optical Fiber With Parabolic Index Profile," B.S.T.J., *52*, No. 7 (September 1973), pp. 1169–1174.
4. D. Marcuse, "Losses and Impulse Response of a Parabolic Index Fiber with Random Bends," B.S.T.J., *52*, No. 8 (October 1973), pp. 1423–1437.
5. D. Marcuse, *Light Transmission Optics*, New York: Van Nostrand Reinhold Co., 1972, p. 270.
6. A. W. Snyder, "Coupled Mode Theory for Optical Fibers," J. Opt. Soc. Am., *62*, No. 11 (November 1972), pp. 1267–1277.
7. D. Marcuse, "Coupled Mode Theory of Round Optical Fibers," B.S.T.J., *52*, No. 6 (July–August 1973), pp. 817–842.
8. D. Marcuse, "Pulse Propagation in Multimode Dielectric Waveguides," B.S.T.J., *51*, No. 6 (July–August 1972), pp. 1199–1232.
9. D. Gloge, "Optical Power Flow in Multimode Fibers," B.S.T.J., *51*, No. 8 (October 1972), pp. 1767–1783.
10. D. Gloge, "Impulse Response of Clad Optical Multimode Fibers," B.S.T.J., *52*, No. 6 (July–August 1973), pp. 801–816.
11. Ref. 5, pp. 272–275.
12. D. Marcuse, "Power Distribution and Radiation Losses in Multimode Dielectric Slab Waveguides," B.S.T.J., *51*, No. 2 (February 1972), pp. 429–454.
13. Ref. 5, p. 96.
14. Ref. 5, p. 104.